Nonlinear dynamics of active particles

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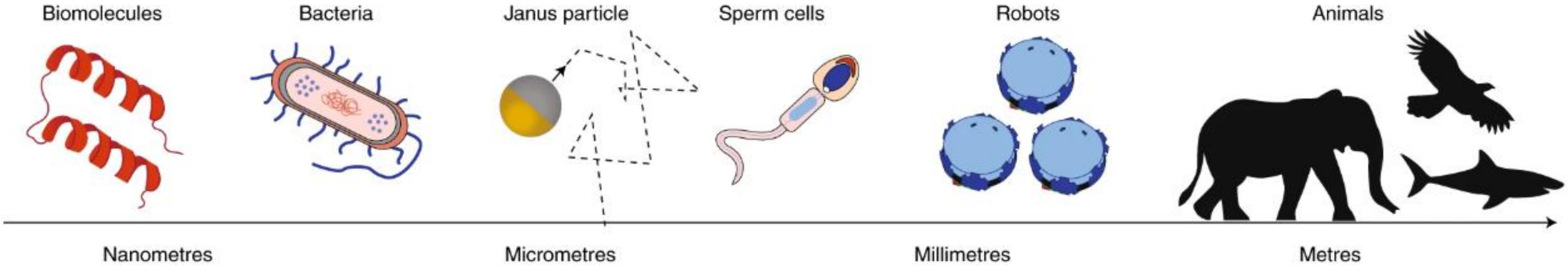
Nonlinear dynamics of active particles

- Active particles and active matter
- Nonlinear dynamical systems
- Nonlinear dynamics of active particles
  - Example 1: Active particle in channel flows
  - Example 2: Superwalking droplets

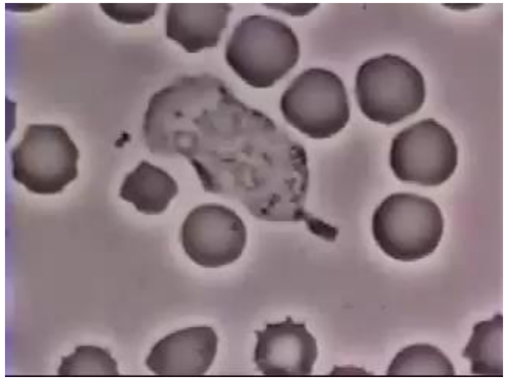
Active particle

- Entity that consume energy and convert it into persistent motion



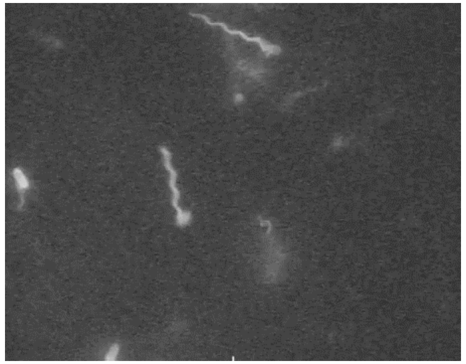
Cichos *et al.* (Nat. Mac. Intell. 2020)

Cells



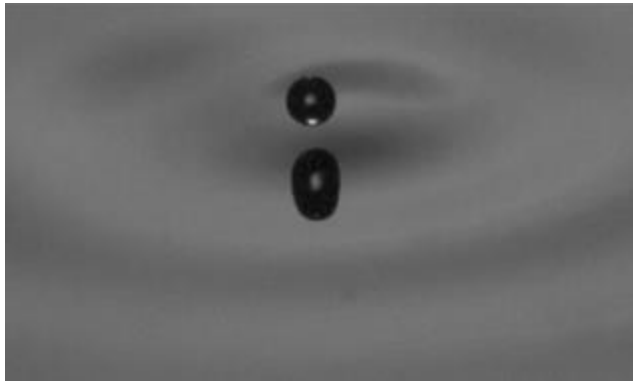
David Rogers (1950s)

Bacteria



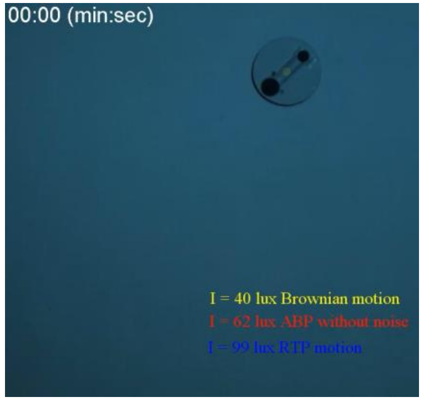
Turner *et al.* (2000)

Walking droplets



Valani *et al.* (2019)

Robots



Paramanick *et al.* (2024)

Active matter

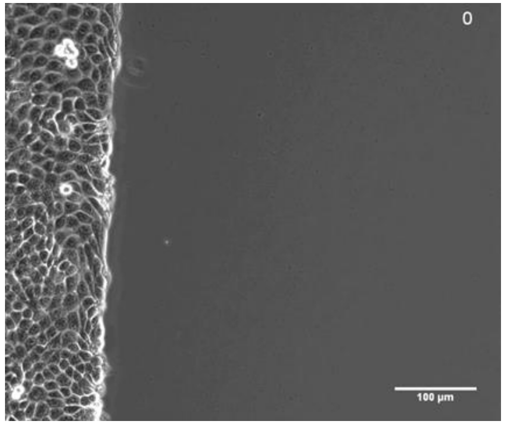
- Matter composed of large number of active particles

- Emergent non-equilibrium behaviors
  - “More is different” (P. Anderson, Science 1972)

Murmuration of birds

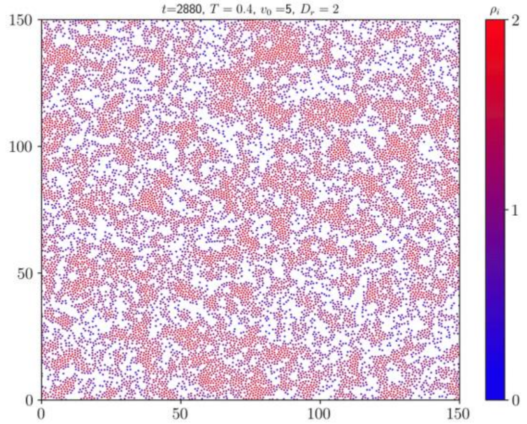
Credit: YouTube

Collective cell migration



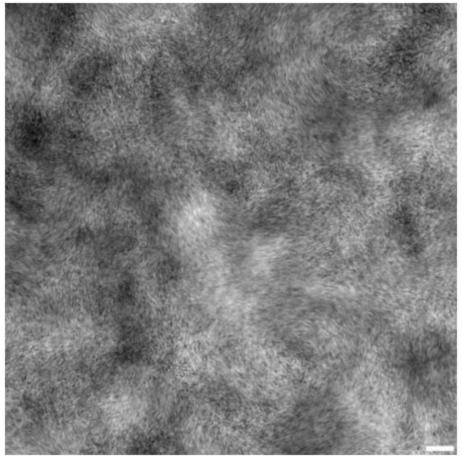
Matsuzawa *et al.* (Cell Rep. 2018)

Motility induced phase separation (MIPS)



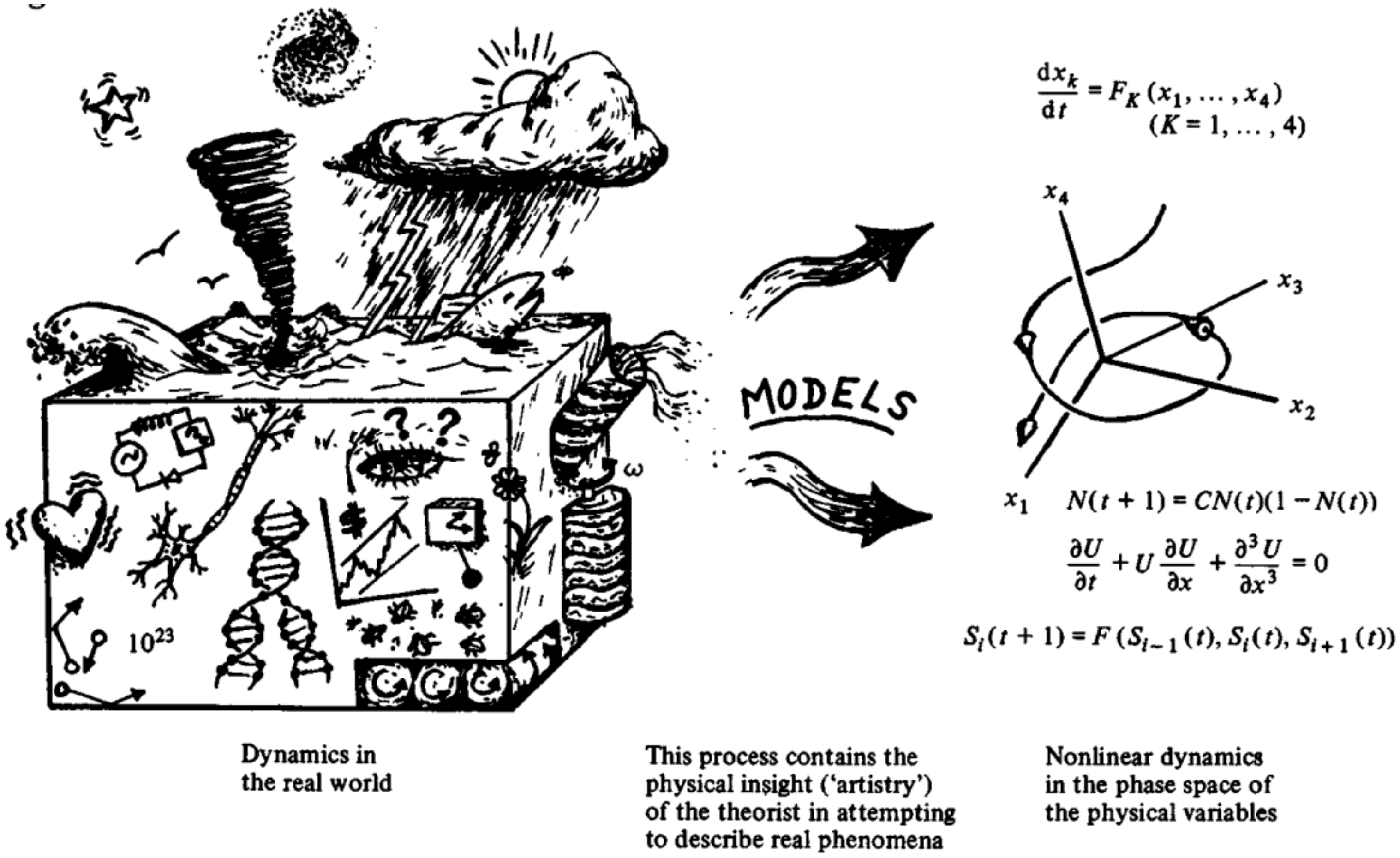
Credit: Gianmarco Spera

Active turbulence



Peng *et al.* (Sci Adv. 2021)

Nonlinear dynamical systems



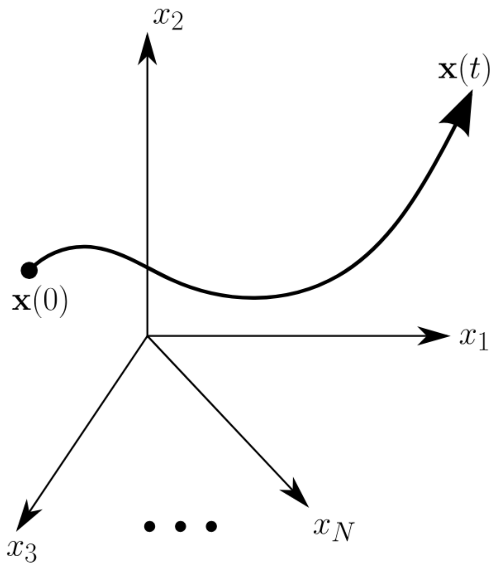
Jackson EA. Perspectives of Nonlinear Dynamics 1. (Cambridge University Press 1990)

$$\mathbf{x} = (x_1, x_2, \dots, x_N)$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

nonlinear functions

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_N) \\ \dot{x}_2 &= f_2(x_1, \dots, x_N) \\ &\vdots \\ \dot{x}_N &= f_N(x_1, \dots, x_N). \end{aligned}$$



Conservative & dissipative dynamical systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

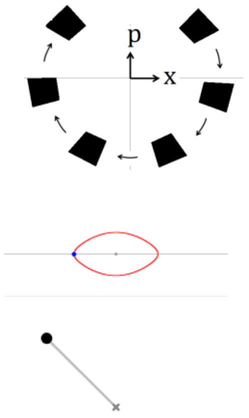
Conservative systems

- Phase-space volume conserved
- Nonlinear systems can exhibit conservative chaos e.g. double pendulum or 3-body problem

Dissipative systems

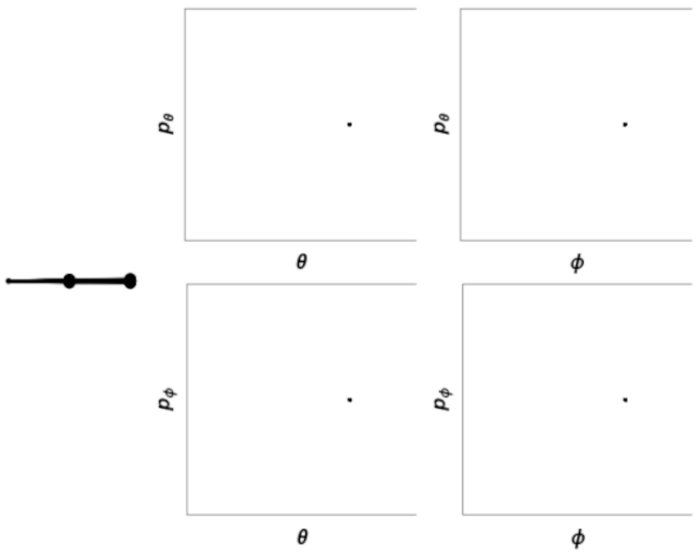
- Phase-space volume shrinks
- Nonlinear systems can exhibit dissipative chaos e.g. Lorenz system

pendulum



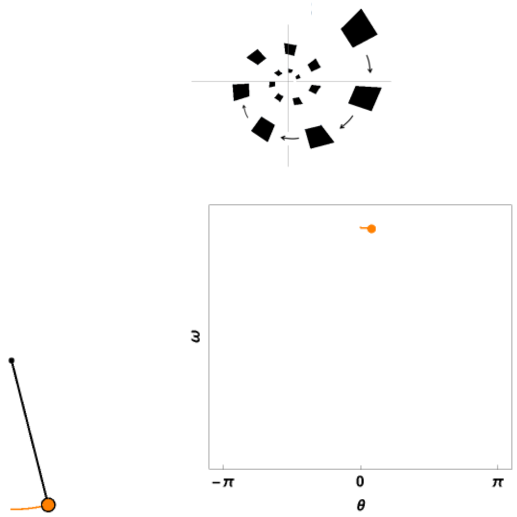
Credit: Lucas Vieira (Wikipedia)

double pendulum



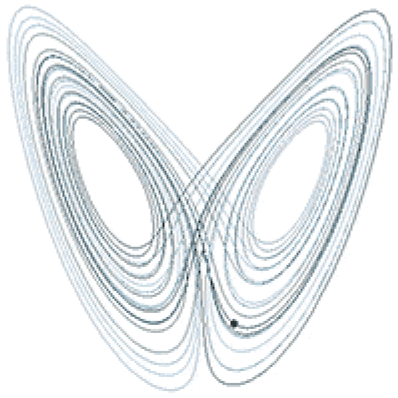
Credit: Jacopo Bertolotti (Wikipedia)

damped pendulum



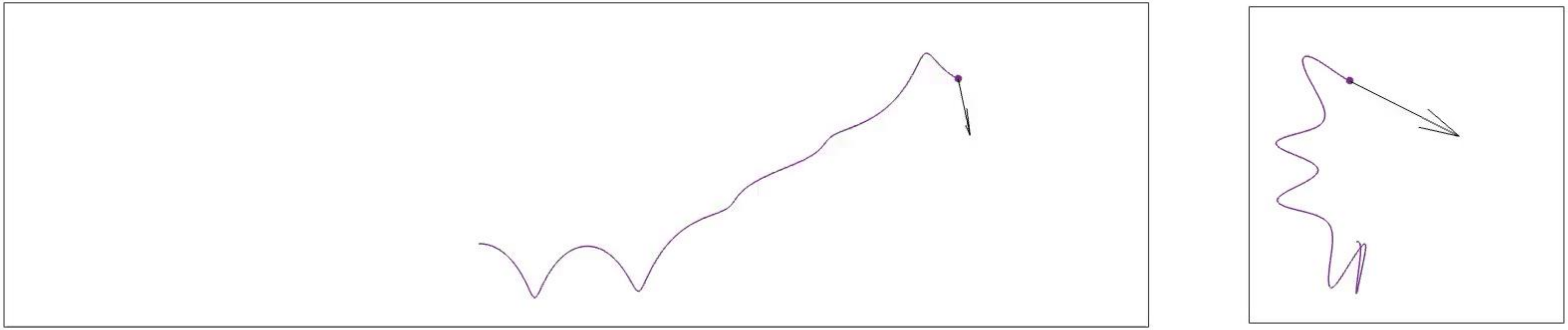
Credit: Jacopo Bertolotti (Wikipedia)

Lorenz chaos



Credit: Dan Quinn (Wikipedia)

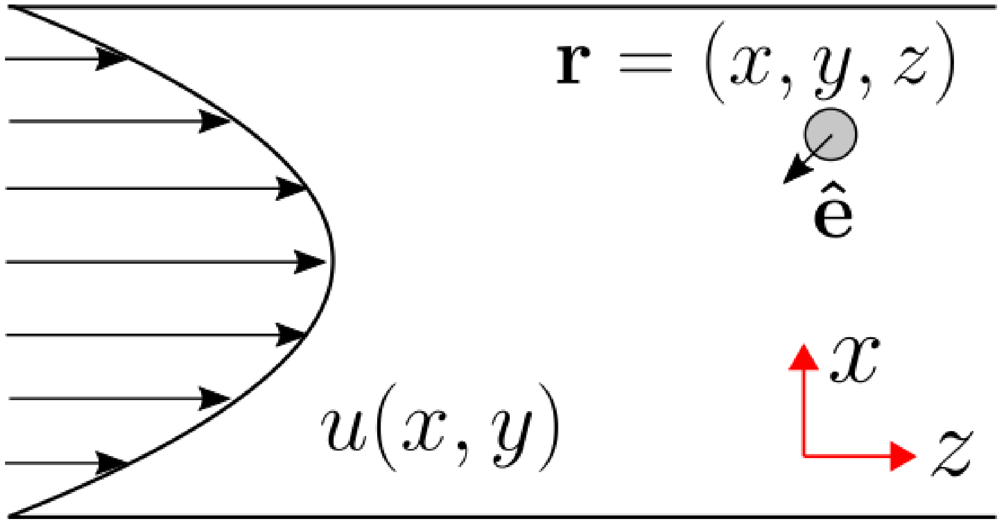
Example 1:
Active particles in unidirectional flows

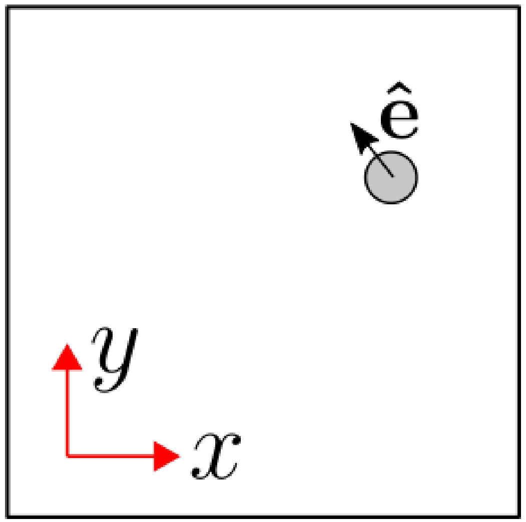


Active particles in unidirectional flows

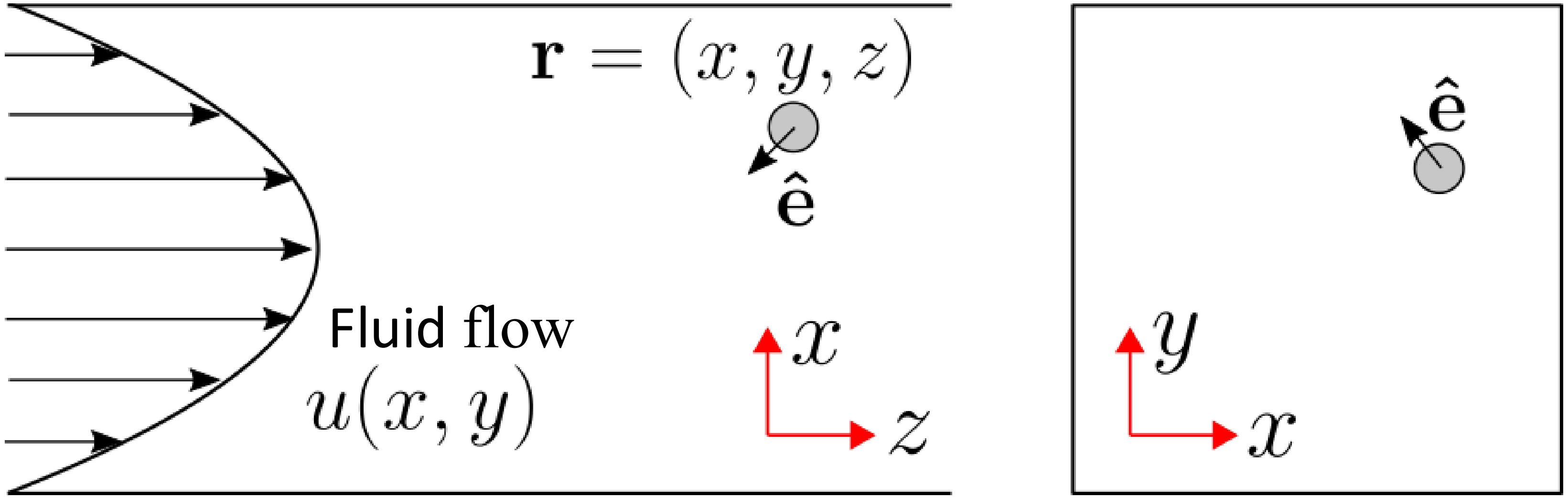
- Microswimmer – active particle immersed in a fluid medium at the microscale e.g. bacteria, motile cell, microrobot

- Microswimmers experience unidirectional fluid flows in confined environments e.g. sperm cells in fallopian tubes, microrobots for targeted drug delivery applications, pathogens in bloodstream





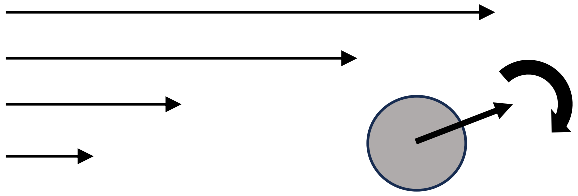
Active particle motion in unidirectional flow



$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = v_0\hat{\mathbf{e}} + u(x,y)\hat{\mathbf{k}}$$

Total velocity = intrinsic (active) velocity + background flow velocity

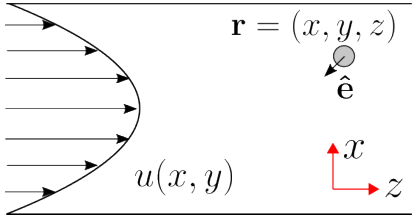
$$\frac{\mathrm{d}\hat{\mathbf{e}}}{\mathrm{d}t} = \frac{1}{2}\left(\nabla \times u(x,y)\hat{\mathbf{k}}\right) \times \hat{\mathbf{e}}$$

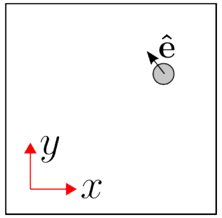


Particle orientation rotates due to local vorticity of fluid

Zöttl & Stark (PRL 2012), Valani, Harding & Stokes (2024 PRE)

Active particle motion in unidirectional flow





- 5 coupled equations
- 2 constants of motion

$$\dot{x} = e_x,\ \dot{y} = e_y,\ \dot{z} = \cancel{e_z +}\ \bar{u}(x,y),$$

$$\dot{e}_x = -\frac{1}{2}e_z\frac{\partial \bar{u}}{\partial x},\ \dot{e}_y = -\frac{1}{2}e_z\frac{\partial \bar{u}}{\partial y},\ \dot{e}_z = \frac{1}{2}e_z\frac{\partial \bar{u}}{\partial x} + \frac{1}{2}e_y\frac{\partial \bar{u}}{\partial y}$$

Constants of motion

$$|\hat{\mathbf{e}}|^2 = e_x^2 + e_y^2 + e_z^2 = 1$$

$$C = -\frac{1}{2}\bar{u}(x,y) + e_z$$

Conservative dynamical system

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x},\ \dot{P}_x = -\frac{\partial \mathcal{H}}{\partial x},\ \dot{y} = \frac{\partial \mathcal{H}}{\partial P_y},\ \dot{P}_y = -\frac{\partial \mathcal{H}}{\partial y}$$

Hamiltonian $\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2) + V_s(x,y)$

Momenta

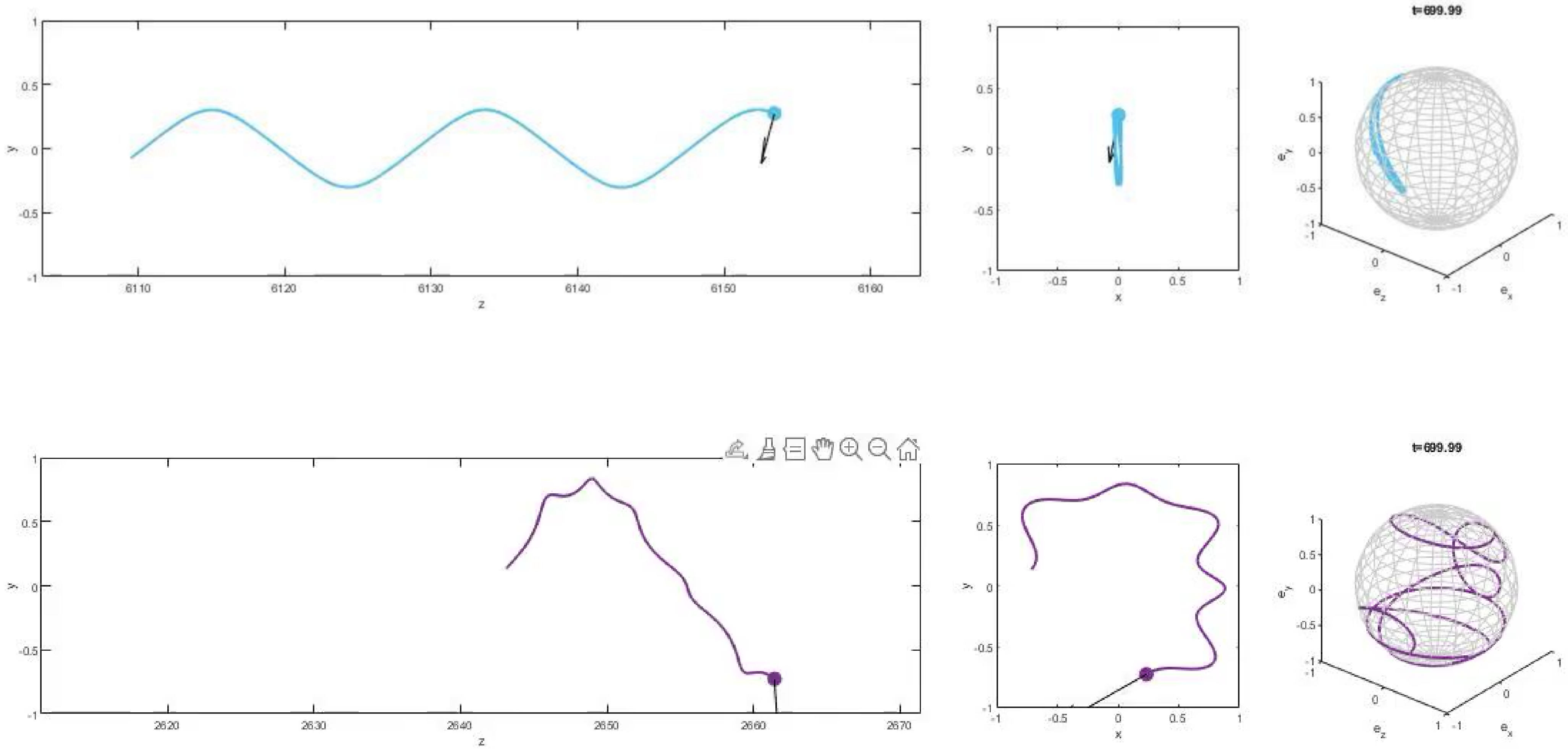
$P_x = \dot{x} \quad P_y = \dot{y}$

Potential energy

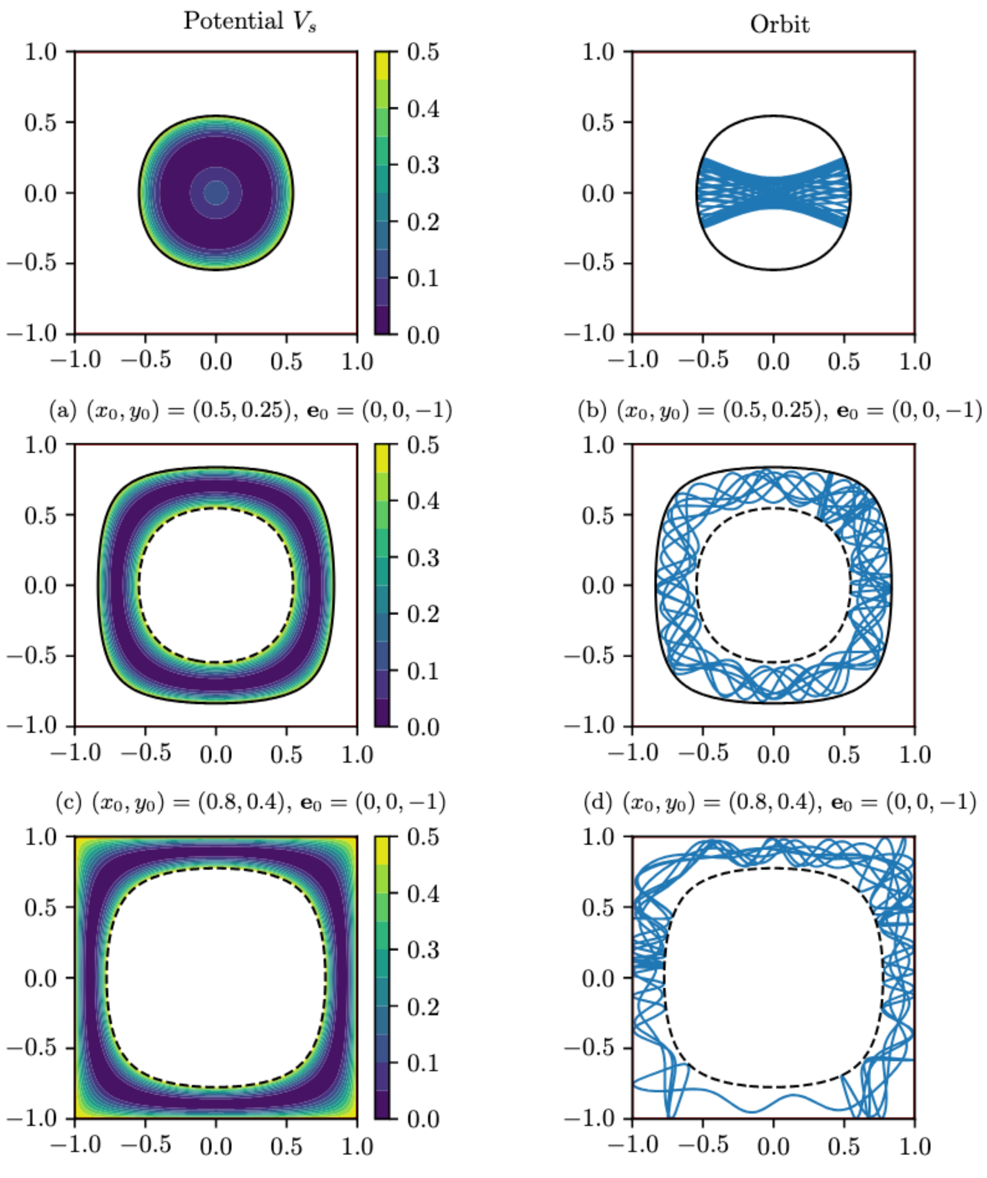
$$V_s = \frac{1}{2}\left(C + \frac{1}{2}\bar{u}(x,y)\right)^2$$

Valani, Harding & Stokes (2024 PRE)
Harding, Valani & Stokes (*in prep.*)

Active particle trajectories in 3D channel with square cross-section



Active particle trajectories in rectangular channels



(a) $(x_0, y_0) = (0.5, 0.25)$, $\mathbf{e}_0 = (0, 0, -1)$ (b) $(x_0, y_0) = (0.5, 0.25)$, $\mathbf{e}_0 = (0, 0, -1)$

(c) $(x_0, y_0) = (0.8, 0.4)$, $\mathbf{e}_0 = (0, 0, -1)$ (d) $(x_0, y_0) = (0.8, 0.4)$, $\mathbf{e}_0 = (0, 0, -1)$

(e) $(x_0, y_0) = (1, 0.5)$, $\mathbf{e}_0 = (0, 0, -1)$ (f) $(x_0, y_0) = (1, 0.5)$, $\mathbf{e}_0 = (0, 0, -1)$

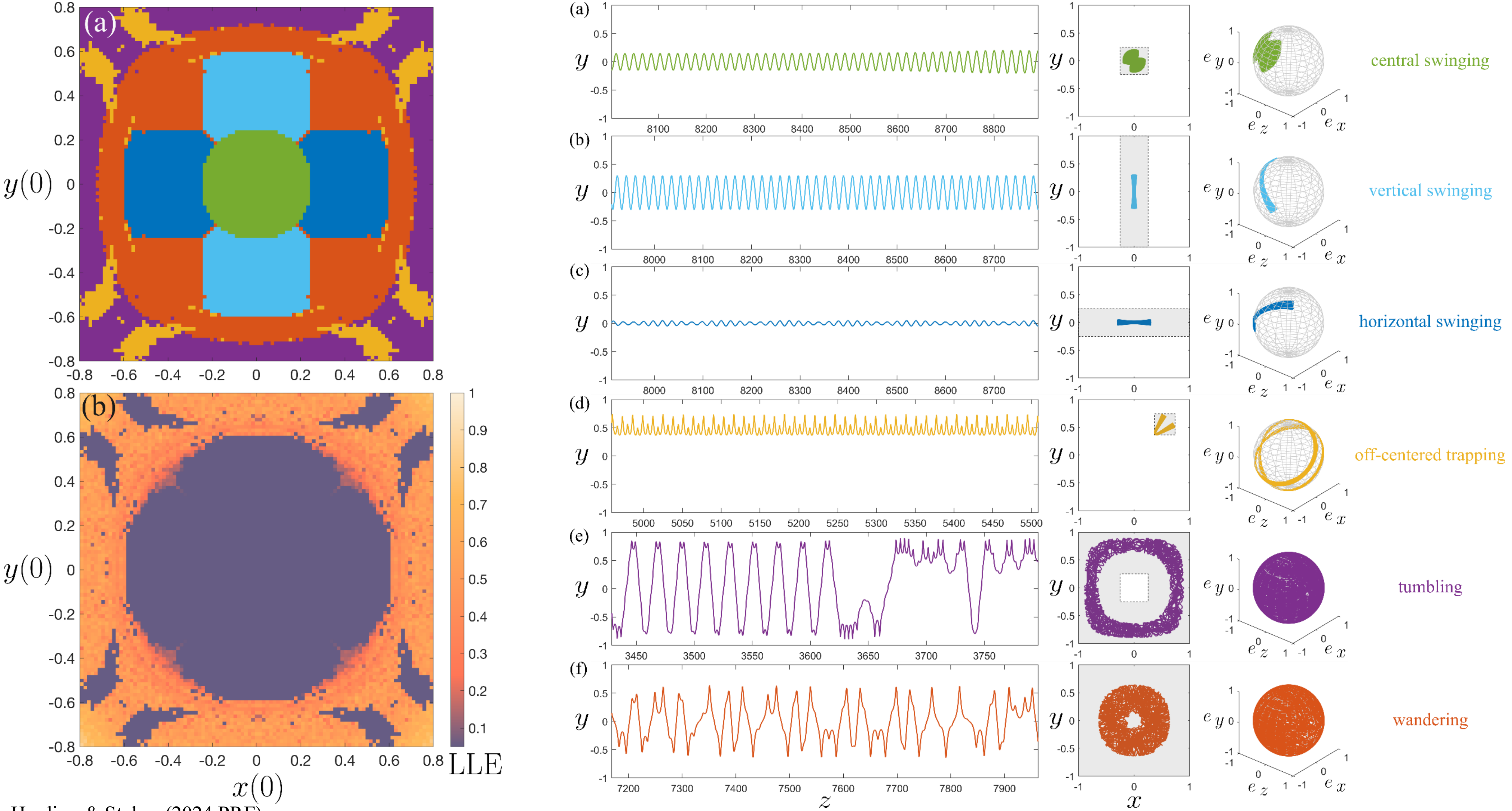
$$C = -\frac{1}{2}\bar{u}(x, y) + e_z$$

Potential energy

$$V_s = \frac{1}{2}\left(C + \frac{1}{2}\bar{u}(x, y)\right)^2$$

Harding, Valani & Stokes (*in prep.*)

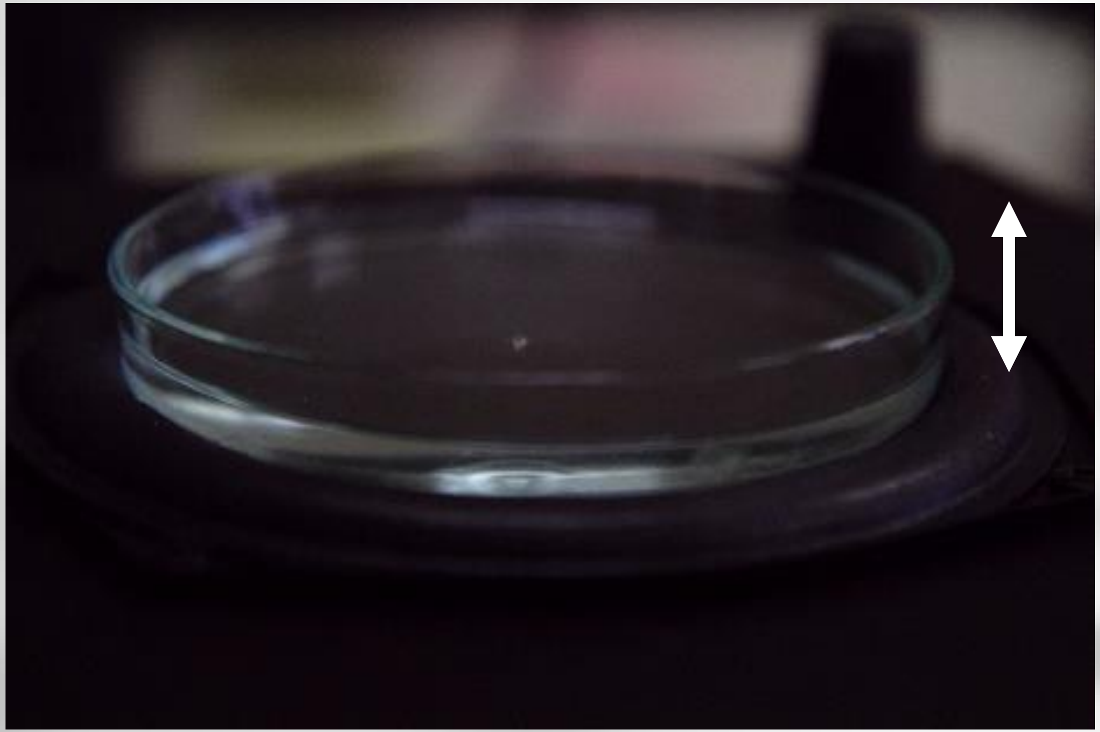
Sensitivity to initial conditions



Valani, Harding & Stokes (2024 PRE)

Example 2:
Superwalking droplets

Flat surface

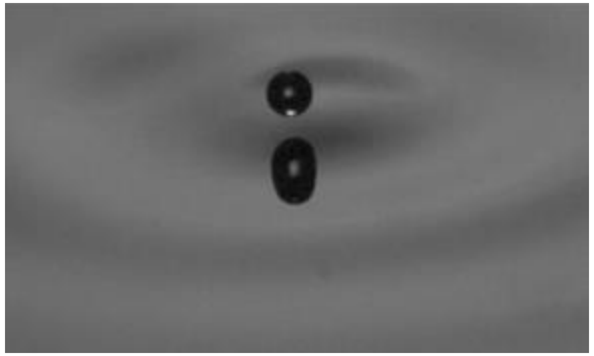


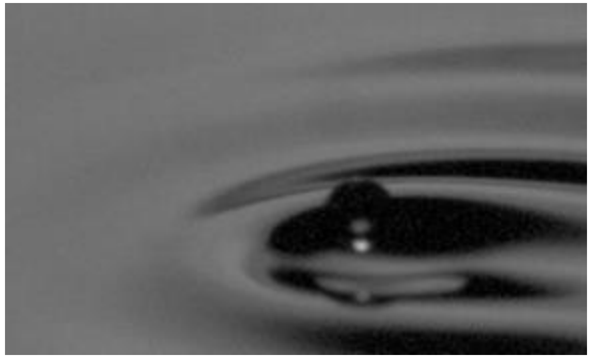
Faraday standing waves

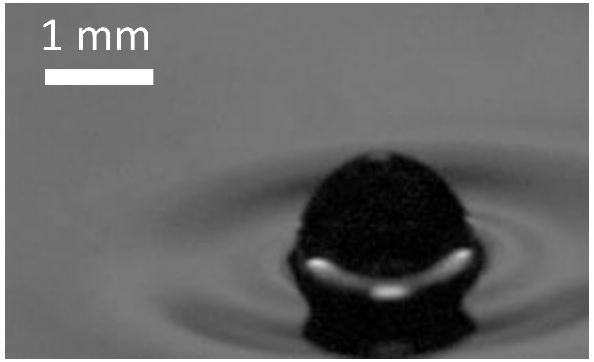
Increasing driving amplitude

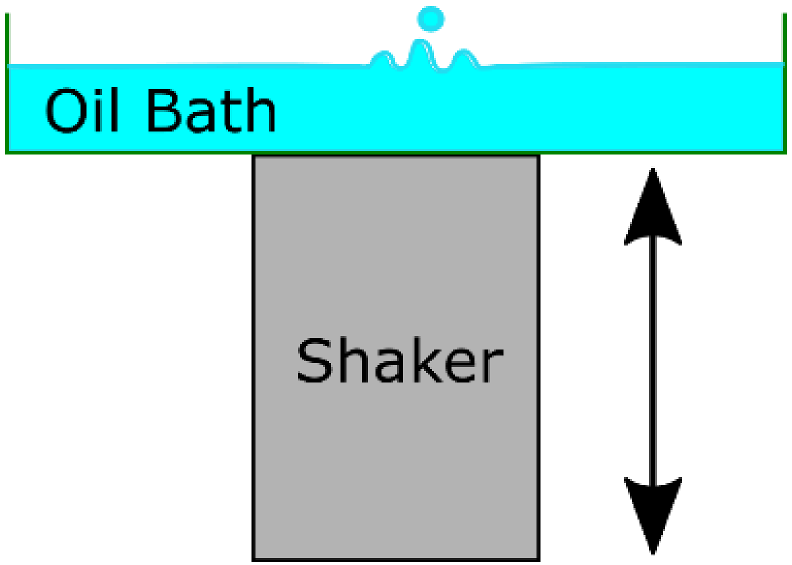
Faraday threshold

Walkers & Superwalkers – active wave-particle entities



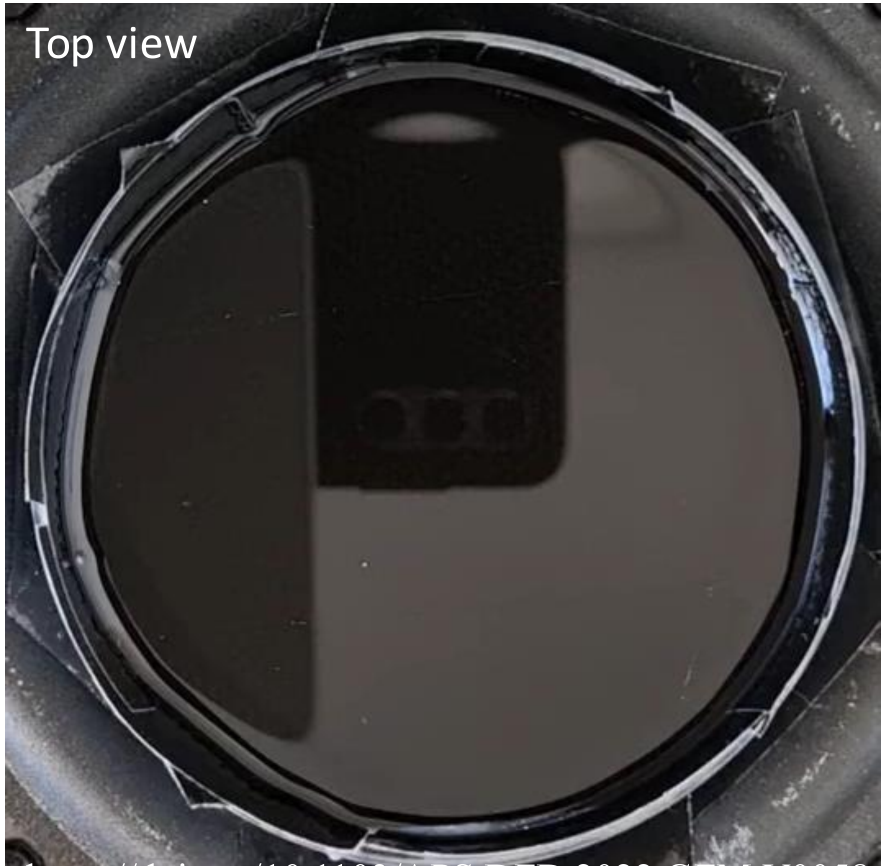






Couder *et al*. (Nature 2005)

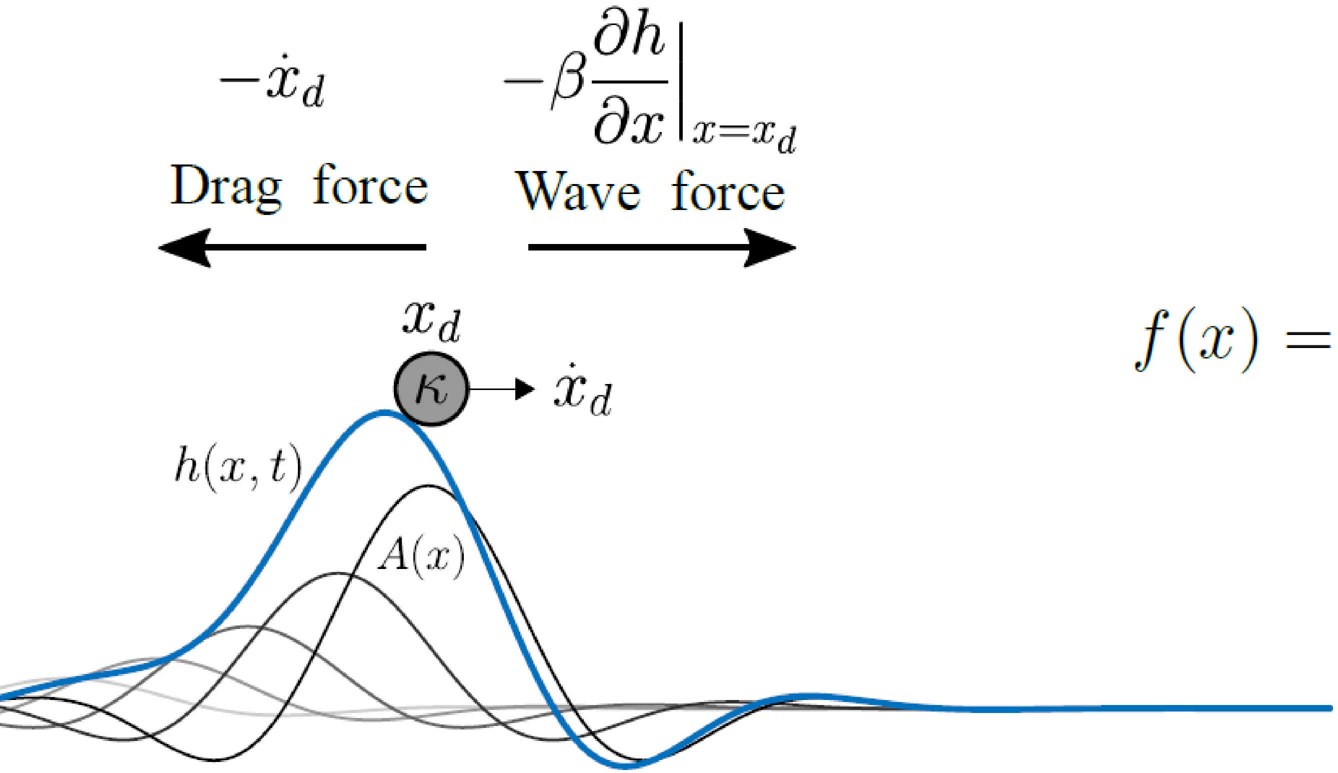
Valani *et al*. (PRL 2019)



https://doi.org/10.1103/APS.DFD.2022.GFM.V0058

Stroboscopic model for the active wave-particle entity

Valani *et al.* (PRL 2019)



$$f(x) = -A'(x)$$

Inertia Drag

$$\kappa \ddot{x}_d + \dot{x}_d = \beta \int_{-\infty}^{t} f(x_d(t) - x_d(s))\, \mathrm{e}^{-(t-s)}\, \mathrm{d}s$$

Wave-memory force

Oza *et al.* (JFM 2013)
Valani *et al.* (PRE 2021)

Connection to Lorenz equations

$$\kappa \ddot{x}_d + \dot{x}_d = \beta \int_{-\infty}^{t} \sin\left(x_d(t) - x_d(s)\right) \mathrm{e}^{-(t-s)}\, \mathrm{d}s$$

$$\sigma = 1/\kappa$$
$$r = \beta$$
$$b = 1$$

<u>Dissipative dynamical system</u>

$$\dot{X} = \sigma\left(Y - X\right)$$
$$\dot{Y} = -Y + rX - XZ$$
$$\dot{Z} = -bZ + XY$$

particle velocity

Wave-memory terms

$$X = \dot{x}_d$$
$$Y = \beta \int_{-\infty}^{t} \sin\left(x_d(t) - x_d(s)\right) \mathrm{e}^{-(t-s)}\, \mathrm{d}s$$
$$Z = \beta \left(1 - \int_{-\infty}^{t} \cos\left(x_d(t) - x_d(s)\right) \mathrm{e}^{-(t-s)}\, \mathrm{d}s\right)$$

Durey (Chaos 2020), Valani *et al.* (PRE 2021), Valani (Chaos 2022)

Dynamics with increasing memory

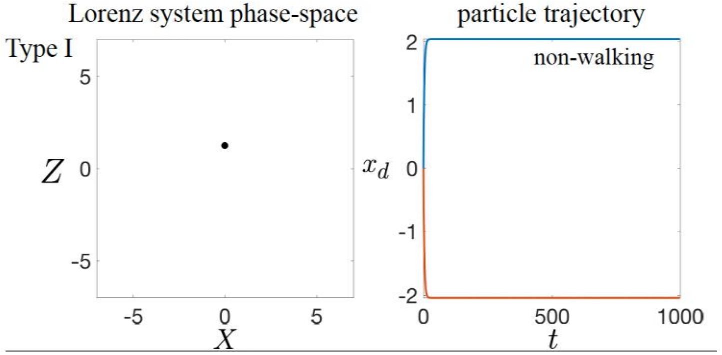
$$\dot{X} = Y - X$$

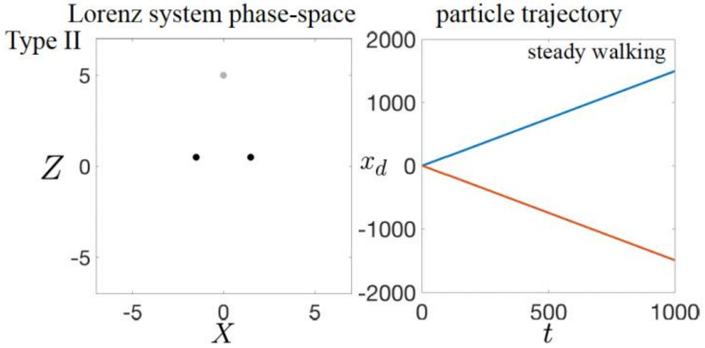
$$\dot{Y} = -\frac{Y}{\tau} + XZ$$

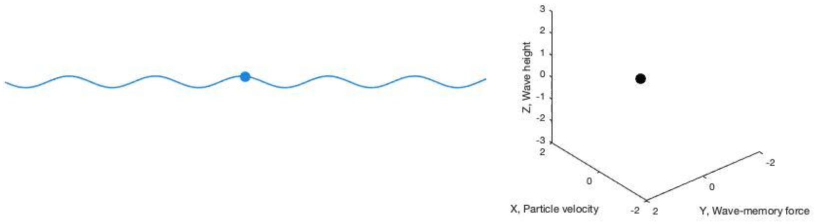
$$\dot{Z} = R - \frac{Z}{\tau} - XY$$

$R =$ dimensionless wave-amplitude

$\tau =$ dimensionless wave-decay rate







stationary state

steady walking state

increasing memory τ

Dynamics with increasing memory

$$\dot{X} = Y - X$$
$$\dot{Y} = -\frac{Y}{\tau} + XZ$$
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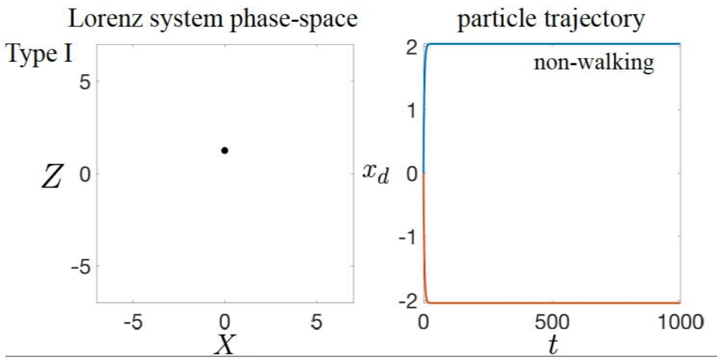
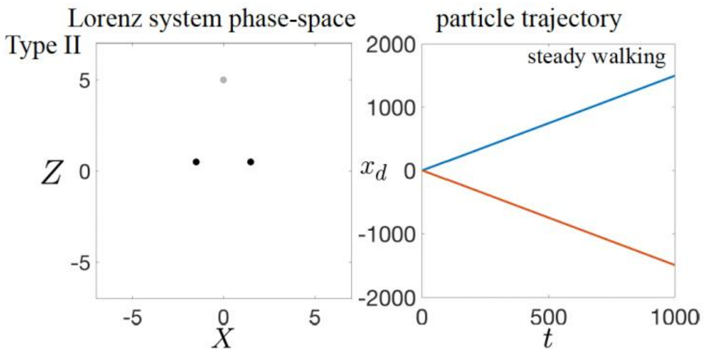
$R =$ dimensionless wave-amplitude

$\tau =$ dimensionless wave-decay rate

Jackson EA. Perspectives of Nonlinear Dynamics 2. (Cambridge University Press 1990)

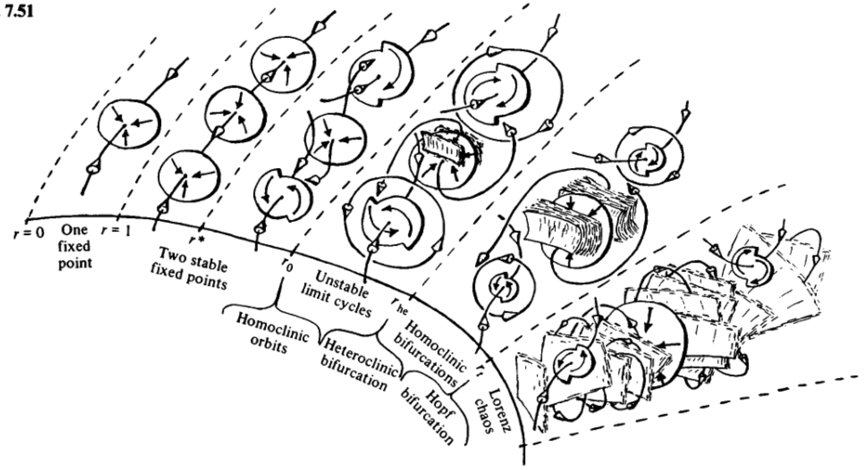


stationary state



steady walking state

Fig. 7.51

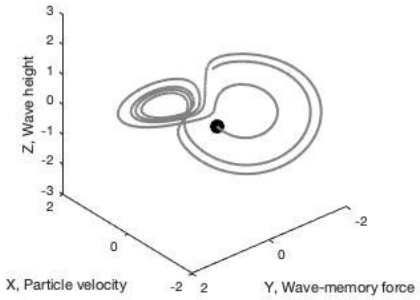


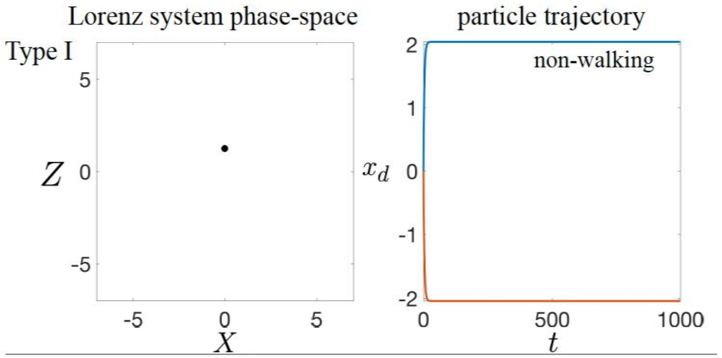
increasing memory τ

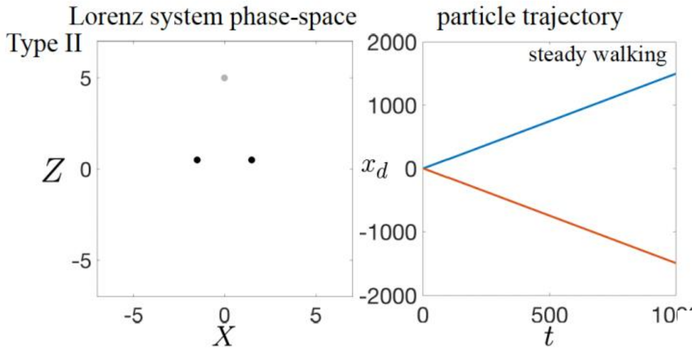
Dynamics with increasing memory

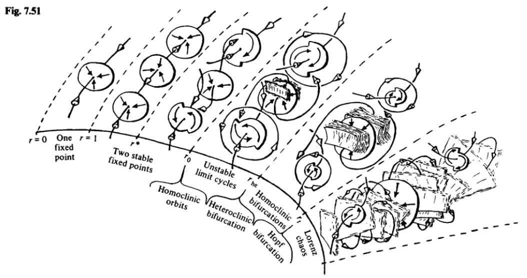
$$\dot{X} = Y - X$$
$$\dot{Y} = -\frac{Y}{\tau} + XZ$$
$$\dot{Z} = R - \frac{Z}{\tau} - XY$$

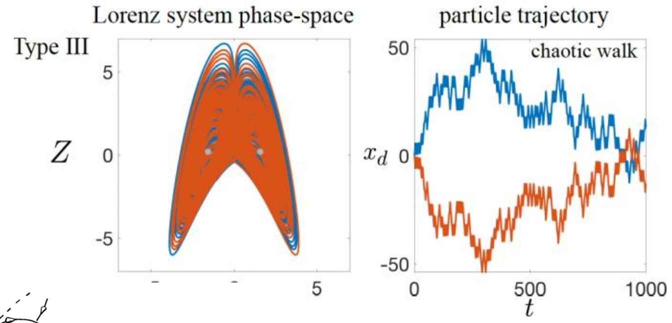
R = dimensionless wave-amplitude
τ = dimensionless wave-decay rate









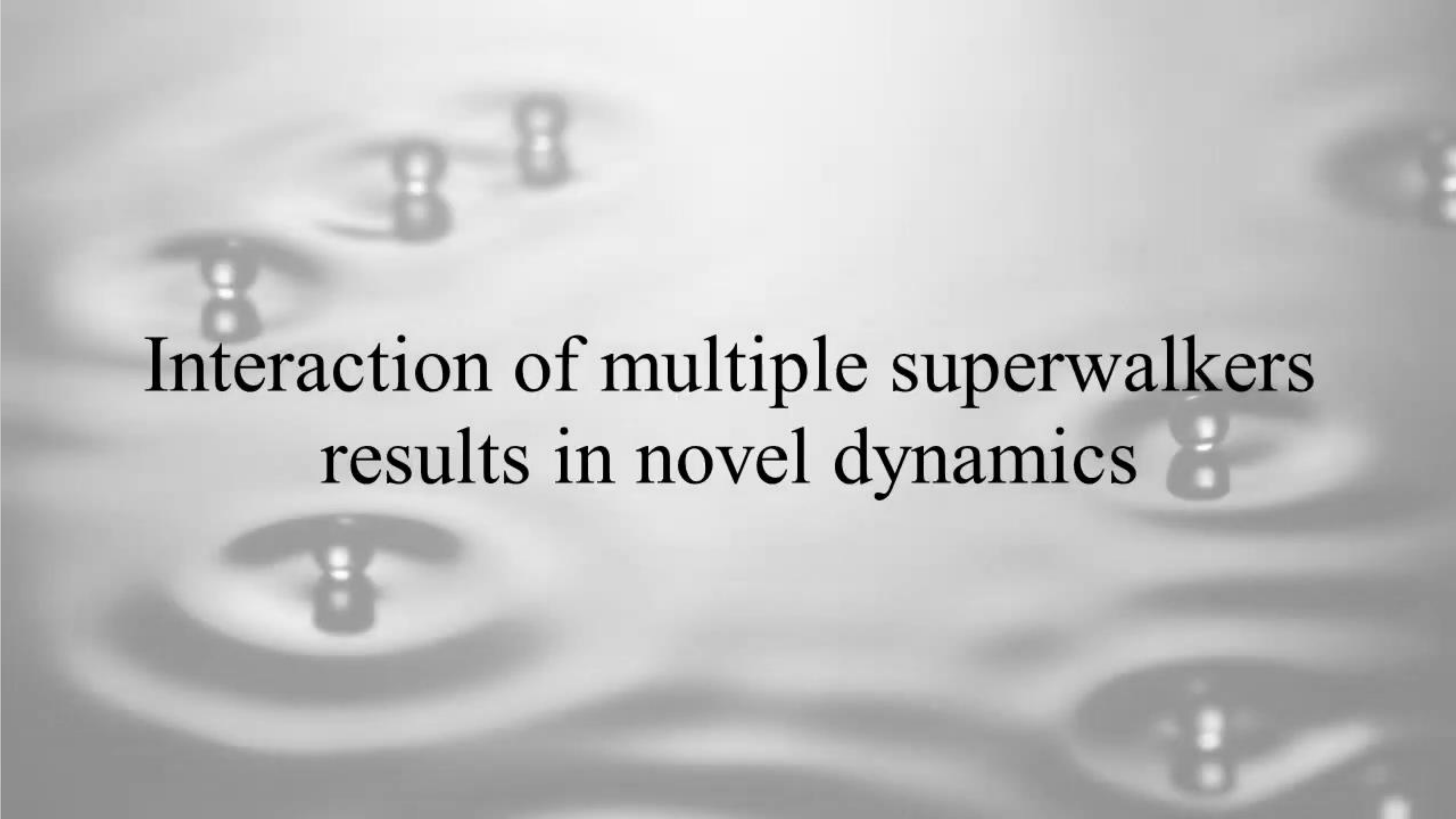


stationary state

steady walking state

chaotic walk

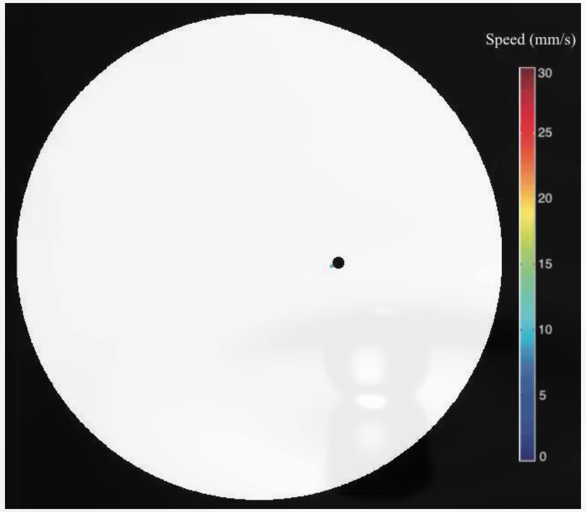
increasing memory τ

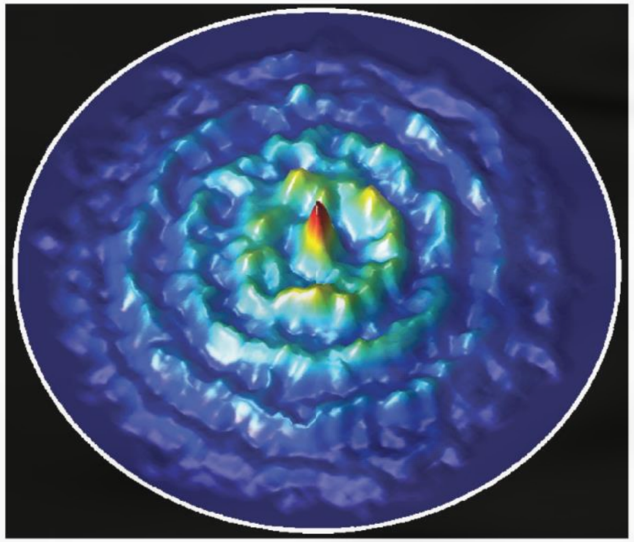


Interaction of multiple superwalkers results in novel dynamics

Hydrodynamic Quantum Analogs

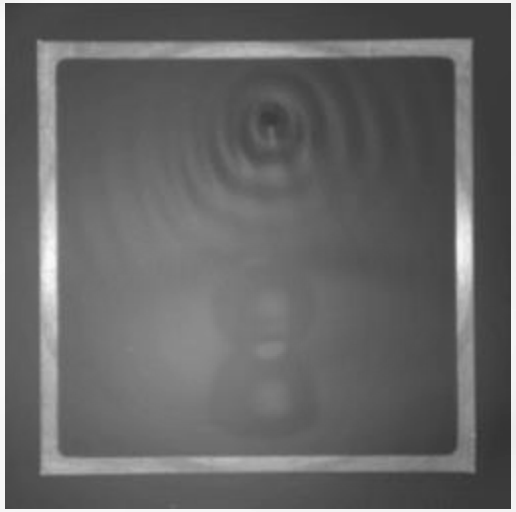
Wave-like statistics in confined geometries



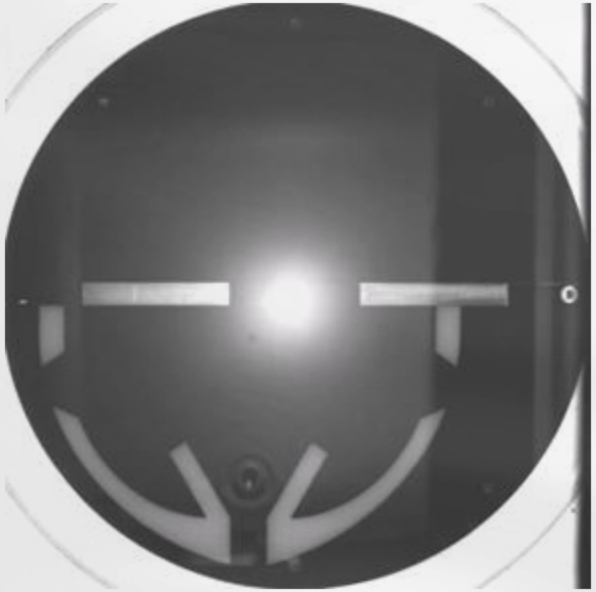


Harris *et al.* (PRE 2013)

Tunnelling across submerged barriers

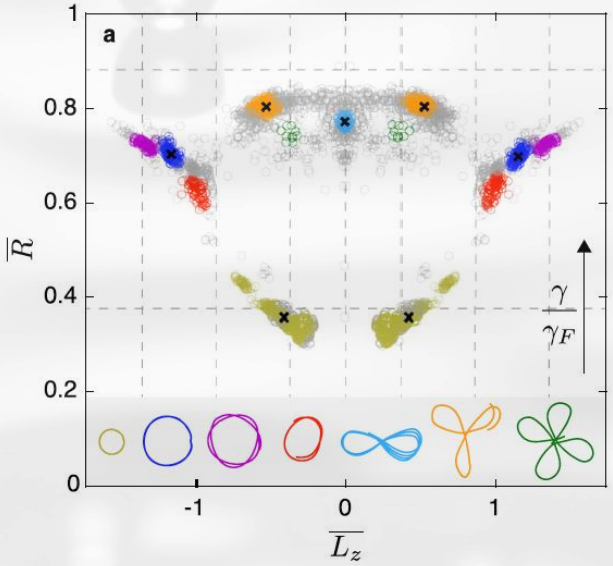


Eddie *et al.* (PRL 2009)



Diffraction & Interference

Couder *et al.* (PRL 2006), Pucci *et al.* (JFM 2013)



Quantisation in droplet’s dynamics

Perrard *et al.* (Nature 2014), Cristea-Platon *et al.* (Chaos 2018)

Summary

- Active particles are non-equilibrium entities that consume energy and convert it to directed motion

- Many interacting active particles result in the emergence of non-equilibrium phases e.g. flocking, active turbulence

- Simple models of active particles can result in nonlinear dynamical systems that exhibit rich complexity of regular and chaotic behaviours

- These dynamical behaviours can be rationalized in terms of conservative and dissipative dynamical systems

Thank you!

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