Sample Solutions

THE COLLEGES OF OXFORD UNIVERSITY

PHYSICS

Wednesday 3 November 2010

Time allowed: 2 hours

For candidates applying for Physics, and Physics and Philosophy

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There are two parts (A and B) to this test, carrying equal weight.

Answers should be written on the question sheet in the spaces provided and you should attempt all the questions. Space for rough working has been left at the end of the paper.

Marks for each question are indicated in the right hand margin. There are a total of 100 marks available and the total marks for each section are indicated at the start of a section. You are advised to divide your time according to the marks available, to spend equal effort on parts A and B, and to attempt **all** the questions on the paper.

No calculators, tables or formula sheets may be used.

Answers in Part A should be given exactly unless indicated otherwise. Numeric answers in Part B should be calculated to 2 significant figures unless otherwise directed.

Use $g = 10\,\mathrm{m\,s^{-2}}$.

Do NOT turn over until told that you may do so.

Part A: Mathematics for Physics [50 Marks]

1. (i) Solve $\sin 3x = \sqrt{3}\cos 3x$ for x in the range $0 \leq x \leq \pi$ [3]

$\Rightarrow \tan 3x = \sqrt{3}$

$\therefore\ 3x = \tan^{-1}(\sqrt{3})$

$= 60° + n \times 180° \quad \text{or} \quad \frac{\pi}{3} + n\pi$

$\therefore\ x = 20° + n \times 60°$

$= 20°, 80°, 140° \quad \text{or} \quad \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$

(ii) Solve $\cos^2 x - \sin x + 1 = 0$ for x also in the range $0 \leq x \leq \pi$ [3]

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

So $1 - \sin^2 x - \sin x + 1 = 0$

$\sin^2 x + \sin x - 2 = 0$ by rearranging

Let $\alpha = \sin x$

$\Rightarrow \alpha^2 + \alpha - 2 = 0$

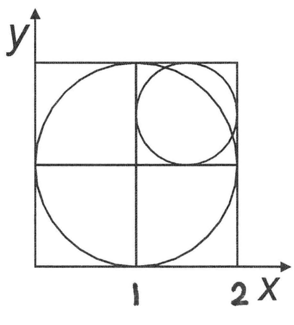
$\therefore\ \alpha = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1$

$\sin x$ can't be -2

$\therefore\ \sin x = 1$

$\therefore\ x = \frac{\pi}{2}$ or $90°$

2. The equation of the larger circle in the figure below is $(x-1)^2 + (y-1)^2 = 1$. Find the equation of the smaller circle. [4]



Centre of big circle is at $(1,1)$

radius $= 1$

$\therefore$ centre of little circle is at $\left(\frac{3}{2}, \frac{3}{2}\right)$

radius $= \frac{1}{2}$

$\therefore$ equation of smaller circle is

$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{1}{4}$

3. Show that $x = -1$ is a root of the polynomial equation $x^3 + 2x^2 - 5x - 6 = 0$, and find the other two roots. [5]

Substituting $x = -1$ into equation gives $-1+2+5-6=0$

$\therefore x=-1$ is a root

$\Rightarrow (x+1)$ is a factor:

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x+1 & x^3+2x^2-5x-6 \\
 & \underline{x^3 + x^2} \\
 & \quad\;\; x^2 - 5x - 6 \\
 & \quad\;\; \underline{x^2 + x} \\
 & \qquad\quad -6x-6 \\
 & \qquad\quad \underline{-6x-6} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\therefore x^3+2x^2-5x-6 = (x+1)(x^2+x-6)$

Solving the quadratic $x^2+x-6 \Rightarrow x = \dfrac{-1 \pm \sqrt{1+24}}{2} = \dfrac{-1 \pm 5}{2} = 2, -3$

$\therefore x^3+2x^2-5x-6 = (x+1)(x-2)(x+3) = 0$

$\therefore$ roots of this equation are $-1, 2, -3$

4. Find the equation of the line passing through the points $A(2, 3)$ and $B(1, 5)$ in the $x - y$ plane. [4]

gradient $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_A - y_B}{x_A - x_B} = \dfrac{3-5}{2-1} = \dfrac{-2}{1} = -2$

$\therefore \; y = -2x + c$

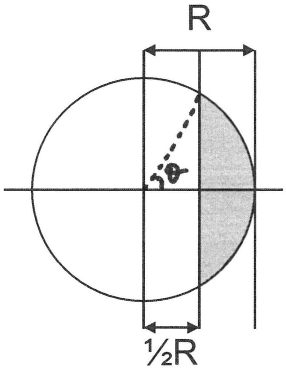
At A, $x=2, y=3$

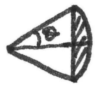
$\Rightarrow 3 = -2 \times 2 + c = -4 + c$

$\Rightarrow c = 7$

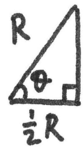
$\therefore \; y = -2x + 7 = 7 - 2x$

5. Find the area of the shaded region of the circle in the figure below, as a function of the radius R.



Shaded area =  − 

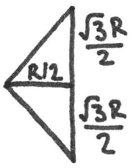
$A = A_1 - A_2$



$\cos\theta = \frac{\frac{1}{2}R}{R} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore A_1 = \frac{\pi R^2}{3}$

[5]



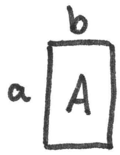
$A_2 = \frac{1}{2} \times \sqrt{3}R \times \frac{R}{2}$

$= \frac{\sqrt{3}R^2}{4}$

$\therefore A = A_1 - A_2$

$= \frac{\pi R^2}{3} - \frac{\sqrt{3}R^2}{4} = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)R^2$

6. A rectangle is formed by bending a length of wire of length L around four pegs. Calculate the area of the largest rectangle which can be formed this way (as a function of L). [4]



$L = 2a + 2b \Rightarrow b = \frac{L}{2} - a$

$A = ab$

$= a\left(\frac{L}{2} - a\right) = \frac{aL}{2} - a^2$

Maximising $A \Rightarrow \frac{dA}{da} = 0 = \frac{L}{2} - 2a \Rightarrow a = \frac{L}{4}$

$\Rightarrow b = \frac{L}{4}$

$\therefore A_{max} = \frac{L^2}{16}$

7. (i) Calculate $\log_3 9$ [2]

$$3^x = 9 \qquad \therefore x = 2$$

(ii) Simplify $\log 4 + \log 16 - \log 2$ [2]

$$= \log 2^2 + \log 2^4 - \log 2$$

$$= 2\log 2 + 4\log 2 - \log 2$$

$$= 5\log 2$$

$$(= \log 32)$$

8. (i) Calculate $(16.1)^2$ [2]

$$\begin{array}{r} 16.1 \\ \times\ 16.1 \\ \hline 1.61 \\ 96.60 \\ 161.00 \\ \hline 259.21 \\ \hline \end{array}$$

(ii) Calculate 10.11×3.2 [2]

$$\begin{array}{r} 10.11 \\ \times\ 3.20 \\ \hline 0 \\ 2.0220 \\ 30.3300 \\ \hline 32.3520 \end{array}$$

$$10.11 \times 3.2 = 32.352$$

9. The first, fourth, and seventh terms of an arithmetic progression are given by x^3, x, and x^2 respectively (where $x \neq 0$ and $x \neq 1$). Find x, and the common difference of the progression. [5]

Terms of progression are $a + (n-1)d$

$n=1 \quad a = x^3 \quad$ Ⓐ

$n=4 \quad a+3d = x \quad$ Ⓑ

$n=7 \quad a+6d = x^2 \quad$ Ⓒ

Ⓑ − Ⓐ $\Rightarrow 3d = x - x^3$

Ⓒ − Ⓑ $\Rightarrow 3d = x^2 - x$

$\therefore x - x^3 = x^2 - x$

$$x - x^3 = x^2 - x$$

$$x(1-x)(1+x) = -x(1-x)$$ Cancel x, $1-x$ (OK because $x \neq 0, x \neq 1$)

$$1+x = -1$$

$$x = -2$$

$$\therefore d = \frac{x^2 - x}{3} = \frac{4+2}{3} = 2$$

10. In a game of dice, a player initially throws a single die, and receives the number of points shown. If the die shows a 6, the player then throws the die again and adds the number shown to his/her score. The player does not throw the die more than twice. Calculate the probability that the player will gain an even number of points. [4]

Throw: 1 2 3 4 5 6

$\downarrow$

1 2 3 4 5 6

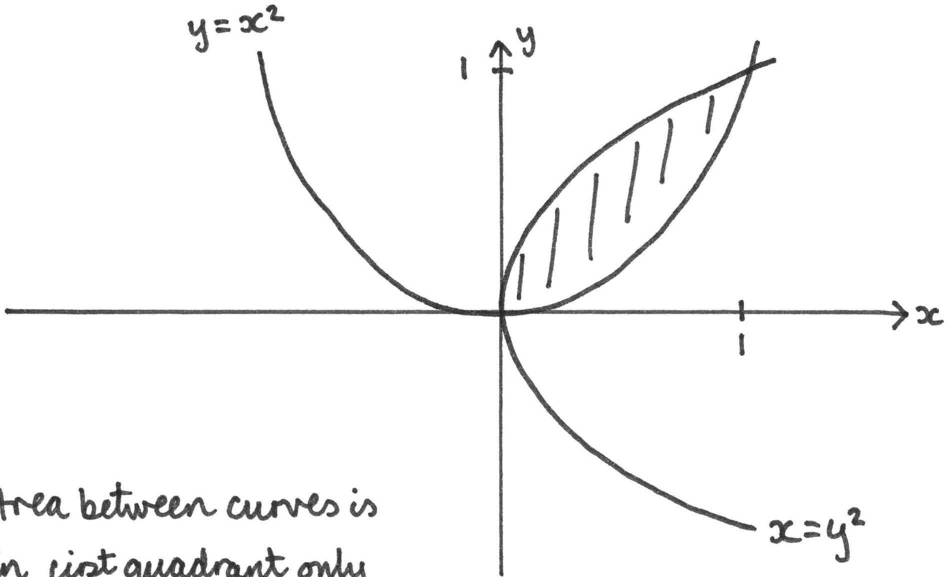
Score: 1 2 3 4 5 7 8 9 10 11 12

Probability: $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{36}$ $\frac{1}{36}$ $\frac{1}{36}$

Total probability of getting even number of points $= \frac{2}{6} + \frac{3}{36} = \frac{15}{36}$

$$= \frac{5}{12}$$

11. Sketch the curves: $y = x^2$ and $x = y^2$, label the points of intersection and calculate the area between the two curves. [5]



Area between curves is in first quadrant only

Curves cross at $x = y = 0$, $x = y = 1$

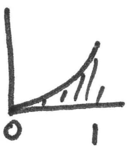
Area under $x = y^2$

$$y = x^{1/2} \quad \therefore \int y\,dx = \int_0^1 x^{1/2}\,dx$$

$$= \left[\frac{2x^{3/2}}{3} \right]_0^1$$

$$= \frac{2}{3}$$

Area under $y = x^2$ $\therefore \int y\,dx = \int_0^1 x^2\,dx$



$$= \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

Area between curves $= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

Part B: Physics [50 Marks]

Multiple choice (10 marks)

Please circle **one** answer to each question only.

12. A rock sample contains two radioactive elements A and B, with half lives of 8000 and 16000 years respectively. If the relative proportion of A:B is initially 1:1, what is their relative proportion after 16000 years?

A	2:1	B	1:2
C	3:1	D	1:3

[1]

13. Two resistors R_1 and R_2 are connected in series with a potential difference V across them. The power dissipated by the resistor R_1 is:

A	$V^2R_1/(R_1+R_2)^2$	B	$V^2{R_2}^2/(R_1(R_1+R_2)^2)$
C	$V^2R_1 \times (R_1+R_2)^2$	D	$V^2{R_2}^2 \times (R_1(R_1+R_2)^2)$

[1]

14. A block of concrete, of mass 100 kg, lies on a 2 m-long plank of wood at a distance 0.5 m from one end. If a builder lifts up the other end of the plank, how much force must he apply to lift the block?

A	125 N	B	12.5 N
C	250 N	D	25 N

[1]

15. A plane flies in a direction NW (according to the plane's internal compass) at an airspeed of 141 km/hr. If the wind at the plane's cruise altitude is blowing with a speed of 100 km/hr directly from the north, what is the plane's actual speed and direction relative to the ground?

A	141 km/hr, SW	B	100 km/hr, W
C	141 km/hr, S	D	223 km/hr, NNW

[1]

16. A teacher wants to listen to a programme on his favourite radio station, broadcasting at a frequency of 1000 kHz, but his radio only indicates the wavelength of the station. To what wavelength must the teacher tune his radio to hear the programme?

A	300 m	B	300000 m
C	0.0033 m	D	50 m

[1]

17. Two mirrors are set at right angles to each other. A beam of light, which is perpendicular to the line of intersection of the two mirrors is incident on the first mirror M_1 at an angle A with respect to its normal. The light reflected by M_1 is then reflected by M_2. What is the angle through which the resultant beam is turned with respect to the incident beam direction?

A Greater than 180° B Exactly 180°
C Less than 180° D It depends on the wavelength [1]

18. A capacitor, of capacitance 3 nF, is charged to a voltage of 10 V. What is the charge held by the capacitor?

A 3.3×10^{9} C B 3×10^{-10} C
C 3×10^{-8} C D 3×10^{-9} C [1]

19. The suspension spring of a car, which has a spring constant of $k = 80000\,\mathrm{Nm^{-1}}$ is sat on by a person weighing 80 kg. By how much is the spring compressed?

A 1 mm B 10 mm
C 5 mm D 20 mm [1]

20. A comet orbits a star. At its closest approach to the star at a distance of 4×10^{10} km, the comet has a speed of 50 km/s. How fast is it travelling when it is at its maximum distance from the star of 10×10^{10} km?

A 50 km/s B 30 km/s
C 20 km/s D 10 km/s [1]

21. A fisherman sees a fish in a river at an apparent depth below the surface of the water of 0.75 m. Given that the refractive index of water is 1.33, is the true depth of the fish below the water's surface:

A 0.75 m?
B Less than 0.75 m?
C 1 m?
D More than 1 m? [1]

Written answers (20 marks)

22. A bathroom contains three rubber ducks, red, green and blue, of identical shape and density, but different overall sizes. The following observations are made:

A) The length of a red duck is equal to the length of a blue duck added to that of a green duck.

B) The area of the base of the green duck is four times larger than the area of the base of the blue duck.

D) The blue duck has a mass of 3 g.

What are the masses of the red and green ducks?

If, when fully submersed, the green duck displaces a total mass of water of 32 g, what is the density of the ducks (the density of water is 1000 kg m^{-3})?

[Hint: Note that for objects of any shape the surface area is proportional to the square of the object's size, and the volume is proportional to the cube of its size.] [6]

A) $r = b + g$ where r, b, g are lengths of red, blue, green ducks

B) $g^2 = 4b^2$ (see Hint)

$\Rightarrow g = 2b$

$\therefore r = 3b$

mass $\propto L^3$

$r^3 = 27b^3 \quad \therefore m_r = 27 \times 3 = 81g$

$g^3 = 8b^3 \quad \therefore m_g = 8 \times 3 = 24g$

$m_b = 3g$

Density $\rho = \frac{m}{V}$

Volume of water displaced has mass 32g

$V = \frac{m_w}{\rho_w} = \frac{32}{1000}$

$\therefore \rho_{DUCK} = \frac{m_{DUCK}}{V} = \frac{24 \times 1000}{32} = 750 \text{ kg/m}^3$

23. Light from the Sun has an approximate flux level of 1×10^3 W m^{-2} at the distance of the Earth and is incident on a steel frying pan of area $0.07\,\mathrm{m}^2$, total mass 2 kg, and initially at a temperature of 20° C. Assuming that the frying pan is perfectly thermally insulated from its surroundings and absorbs all the sunlight incident upon it, how long does it take for the pan to reach a temperature of 70° C and thus be hot enough to fry an egg? [3]

The frying pan, still at 70° C, is then plunged into a bowl containing 4 kg of water at 20° C. Assuming the bowl has negligible heat capacity and assuming that there is no heat flow to or from the surroundings, what is the final temperature of the water in the bowl to the nearest °C? [4]

(The specific heat capacity of steel is 490 J kg^{-1} K^{-1} and the specific heat capacity of water is 4200 J kg^{-1} K^{-1}).

[Hint: For the second part of the question, you may find it easier to write the problem in terms of the temperature change of the water.]

a) Power received by pan = flux x area

$= 1000 \times 0.07 = 70\,\mathrm{W}$

Time to heat up is Δt

$$P\Delta t = mC\Delta T \Rightarrow \Delta t = \frac{mC\Delta T}{P}$$

$$= \frac{2 \times 490 \times 50}{70}$$

$$= 2 \times 7 \times 50 = 700\,\mathrm{s}$$

b) Final temperature of pan + water $= T_1$

Heat flow from pan $= 2 \times 490 \times (70 - T_1)$

Heat flow into water $= 4 \times 4200 \times (T_1 - 20)$

Let $\Delta T = T_1 - 20 \Rightarrow T_1 = \Delta T + 20$ so $70 - T_1 = 50 - \Delta T$

Heat flows balance

$$\Rightarrow 2 \times 490 \times (50 - \Delta T) = 4 \times 4200 \times \Delta T$$

$$49 \times (50 - \Delta T) = 840\Delta T$$

$$49 \times 50 - 49\Delta T = 840\Delta T$$

$$2450 = 889\Delta T$$

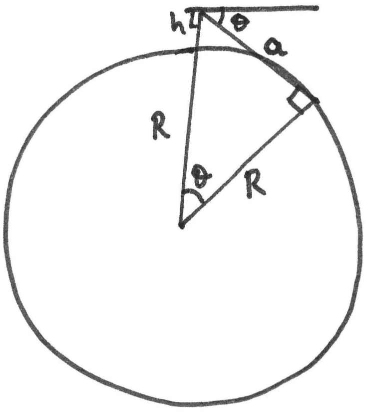
$$\Delta T = \frac{2450}{889} \simeq 3°\mathrm{C}$$

$\therefore T_{final} = 20 + 3 = 23°\mathrm{C}$

24. An astronaut arrives on the planet Oceania and climbs to the top of a cliff overlooking the sea. The astronaut's eye is 100 m above the sea level and he observes that the horizon in all directions appears to be at angle of 5 mrad below the local horizontal. What is the radius of the planet Oceania at sea level? [4]

How far away is the horizon from the astronaut? [3]

[Hint: the line of sight from the astronaut to the horizon is tangential to surface of the planet at sea level.]



a is astronaut's line of sight (distance to horizon)

Tangent meets radius at right-angle

$$\therefore (R+h)^2 = R^2 + a^2$$

$$R^2 + 2Rh + h^2 = R^2 + a^2$$

$$2Rh \sim a^2 \quad (h^2 \text{ small})$$

$$\therefore a \sim \sqrt{2Rh}$$

$$\sin\theta = \frac{a}{R+h} \sim \frac{a}{R} = \sqrt{\frac{2h}{R}}$$

$$\sin^2\theta = \frac{2h}{R}$$

$$(5\times10^{-3})^2 = \frac{200}{R}$$

$$\therefore R = \frac{200}{25\times10^{-6}} = \frac{800}{100\times10^{-6}} = 8\times10^6\,\text{m}$$

$$= 8000\,\text{km}$$

$$a = \sqrt{2Rh} = \sqrt{2\times8000\times0.1}$$

$$= \sqrt{1600}$$

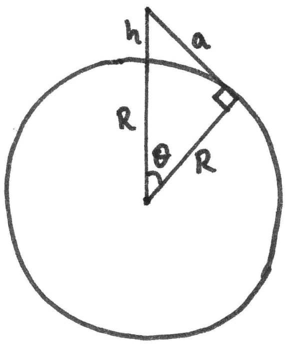
$$= 40\,\text{km}$$

ALTERNATIVE APPROACH

24. An astronaut arrives on the planet Oceania and climbs to the top of a cliff overlooking the sea. The astronaut's eye is 100 m above the sea level and he observes that the horizon in all directions appears to be at angle of 5 mrad below the local horizontal. What is the radius of the planet Oceania at sea level? [4]

How far away is the horizon from the astronaut? [3]

[Hint: the line of sight from the astronaut to the horizon is tangential to surface of the planet at sea level.]



a is astronaut's line of sight (distance to horizon)

Tangent meets radius at right-angle

$$\cos\theta = \frac{R}{R+h} \quad \Rightarrow (R+h)\cos\theta = R$$

$$\cos\theta \sim \left(1 - \frac{\theta^2}{2}\right)$$

$$\therefore (R+h)\left(1-\frac{\theta^2}{2}\right) = R$$

$$R+h = R\left(1-\frac{\theta^2}{2}\right)^{-1}$$

$$\sim R\left(1+\frac{\theta^2}{2}\right)$$

$$R+h = R + \frac{R\theta^2}{2}$$

$$\therefore R = \frac{2h}{\theta^2} = \frac{200}{(5\times10^{-3})^2}$$

$$= \frac{200}{25\times10^{-6}} = \frac{800}{100\times10^{-6}}$$

$$= 8\times10^6\,\text{m} = 8000\,\text{km}$$

Distance to horizon:

$$(R+h)^2 = R^2 + a^2$$

$$2Rh + h^2 = a^2$$

$$2Rh \sim a^2$$

$$a = \sqrt{2Rh}$$

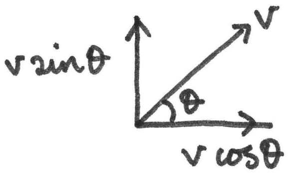
$$= \sqrt{2\times 8000 \times 0{\cdot}1}$$

$$= \sqrt{1600} = 40\,\text{km}$$

Long question (20 marks)

25. A gun is designed that can launch a projectile, of mass 10 kg, at a speed of 200 m/s. The gun is placed close to a straight, horizontal railway line and aligned such the projectile will land further down the line. A small rail car, of mass 200 kg and travelling at a speed of 100 m/s passes the gun just as it is fired. Assuming the gun and the car are at the same level, at what angle upwards must the projectile be fired in order that it lands in the rail car?

[3]



To land in car, match car speed

$v \cos\theta = 100$ $\qquad v = 200\,m/s$

$\therefore \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = 60°$

How long does it take for the projectile to reach its maximum altitude? [3]
(You may use $\sqrt{3} \approx 1.732$)

$V_{\uparrow} = v \sin\theta = 200 \times \frac{\sqrt{3}}{2} = 100\sqrt{3}\ m/s$

$v = u + at_1$

At max altitude:

$0 = 100\sqrt{3} - 10t_1 \implies t_1 = 10\sqrt{3}\,s$

$= 17\,s \ (2sf)$

How far is the rail car from the gun when the projectile lands in it? [3]

$t_{land} = 2 \times t_1 = 20\sqrt{3}\,s$

$\therefore D = V_{\rightarrow} \times t_{land}$

$= 100 \times 20\sqrt{3}$

$= 2000\sqrt{3}$

$= 3500\,m \ (2sf)$

Without considering energy, calculate the projectile's maximum altitude. [2]

$$s = ut + \frac{1}{2}at^2$$

$$= 100\sqrt{3} \times 10\sqrt{3} - \frac{1}{2} \times 10 \times 100 \times 3$$

$$= 3000 - 1500$$

$$= 1500\,m$$

Now consider energy. What is the initial kinetic energy of the rail car and what is the initial kinetic energy of the projectile in both the vertical and horizontal directions? [3]

$$K_{car} = \frac{1}{2} M_c V_c^2 = \frac{1}{2} \times 200 \times 100^2 = 10^6\,J$$

$$K_p^{\uparrow} = \frac{1}{2} m_p V_{\uparrow}^2 = \frac{1}{2} \times 10 \times 100^2 \times 3 = 1.5 \times 10^5\,J$$

$$K_p^{\rightarrow} = \frac{1}{2} m_p V_{\rightarrow}^2 = \frac{1}{2} \times 10 \times 100^2 = 5 \times 10^4\,J$$

Using your calculation of the projectile's initial kinetic energy, again calculate the projectile's maximum altitude. [2]

$$mgh = \frac{1}{2} m_p V_{\uparrow}^2$$

$$10 \times 10 \times h = 1.5 \times 10^5$$

$$h = 1.5 \times 10^3 = 1500\,m$$

When the projectile lands in the rail car, why does the velocity of the car not change? [2]

Projectile has same horizontal velocity as car

so no transfer of momentum.

(Vertical energy gets dissipated as heat)

Assuming that the projectile remains in the car, what is the combined kinetic energy of the car plus projectile after the projectile has landed (to three significant figures)? [2].

$$K_{c+p} = \frac{1}{2} m_p v_{\rightarrow}^2 + K_{car}$$

$$= 100 \times 10^4 + 5 \times 10^4$$

$$= 105 \times 10^4 \text{ J}$$